Green function estimates for second order elliptic operators in non-divergence form

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The 16th International Workshop on Markov Processes and Related Topics

12-16 July, 2021

Joint work with Professor Zhen-Qing Chen

Problem:

- Consider the following elliptic operator in non-divergence form on $\mathbb{R}^d$ with $d \geq 3$

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x).$$

- where $(a_{ij}(x))_{1 \leq i,j \leq d}$ is a symmetric $d \times d$ matrix-valued uniformly elliptic function on $\mathbb{R}^d$ and satisfies the Dini continuity condition.
- Question: What are the two-sided estimates of the $\mathcal{L}$-Green function $G_D(x,y)$ on a bounded $C^{1,1}$ domain D as well as its first and second order derivatives estimates?

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- Question: What are the two-sided estimates of the $\mathcal{L}$-Green function $G_D(x,y)$ on a bounded $C^{1,1}$ domain D as well as its first and second order derivatives estimates?

Background: Green function for Δ on bounded $C^{1,1}$ domain

- Probabilistic definition. The Green function $G_D^\Delta(x,y)$ is the occupation density of Brownian motion W on D. That is, for each $f \in C_c^\infty(D)$,

$$\mathbb{E}_x \int_0^{\tau_D} f(W_s)\, ds = \int_D G_D^\Delta(x,y) f(y)\, dy.$$

- Analytic definition. Suppose D is a bounded domain and the boundary ∂D is regular. Suppose φ is locally Hölder continuous in D. Then

$$u(x) := G_D^\Delta \varphi(x) = \int_D G_D^\Delta(x,y)\varphi(y)\, dy$$

is the unique solution of the following Poisson equation

$$\begin{cases} \Delta u &= -\varphi, \quad \text{in} \quad D \\ u &= 0, \quad \text{on} \quad \partial D. \end{cases}$$

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Suppose D is a bounded $C^{1,1}$ domain with characteristics (R_0, Λ_0) in $\mathbb{R}^d$ with $d \geq 3$.

- Widman(1967) and Zhao(1986) proved that there is a constant $C > 1$ depending on $(d, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for $x \neq y$ in D,

$$\frac{C^{-1}}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \leq G_D^{\Delta}(x,y)$$
$$\leq \frac{C}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right).$$

- For each $y \in D$, $G_D^{\Delta}(\cdot, y)$ is Δ-harmonic in $D \setminus \{y\}$. By the classical harmonic function theory, $G_D^{\Delta}(x,y)$ is a smooth function for $(x,y) \in D \times D \setminus diag$.

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By the derivative estimates of Δ-harmonic function in balls, there exists a constant $C = C(d, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $x \neq y$ in D,

$$|\nabla_x G_D^\Delta(x,y)| \le \frac{C}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right);$$

and

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G_D^\Delta(x,y)\right| \le \frac{C}{|x-y|^d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{-1}.$$

Background: Divergence form elliptic operator

- Let

$$\mathcal{L}^0 f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) f(x).$$

be a uniformly elliptic operator with uniform ellipticity constant λ_0.

- Grüter-Widman(1982) proved that when D is a bounded domain, there exists a unique function $G_D^0(x,y)$ such that for each $\phi \in C_c^\infty(D)$,

$$-(\mathcal{L}^0 G_D^0(\cdot, y), \phi) = \phi(y).$$

Moreover,

$$G_D^0(x,y) \le C(d,\lambda_0)|x-y|^{2-d}, \quad x \ne y \in D.$$

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$$G_D^0(x,y) \le C(d,\lambda_0)|x-y|^{2-d}, \quad x \neq y \in D.$$

- Furthermore, Grüter-Widman(1982) proved when D satisfies the exterior sphere condition and suppose a_{ij} satisfies the Dini condition, that is,

$$|a_{ij}(x) - a_{ij}(y)| \leq \ell(|x-y|),$$

where $\ell(\cdot) : [0,\infty) \to [0,\infty)$ is a monotonically increasing continuous function with $\ell(0) = 0$ and

$$\int_0^1 \ell(t)/t \, dt < \infty.$$

Then there is a constant $C = C(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for $x \neq y$ in D,

$$G_D^0(x,y) \leq \frac{C^{-1}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)$$

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Background: Non-divergence form elliptic operator

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$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x).$$

- Unlike the case of divergence form operators that is uniformly elliptic and bounded, Bauman(1986) showed a counterexample that $G_D(x,y)$ for a bounded and smooth domain D may not be bounded by $c|x-y|^{2-d}$ even when the coefficient a_{ij} is uniformly continuous.
- When a_{ij}, $1 \leq i,j \leq d$ are Hölder continuous, Hueber and Sieveking(1982) proved that the Green function of $\mathcal{L}$ on bounded $C^{1,1}$ domain is comparable to the Green function of Δ.

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- Suppose a_{ij} satisfies the Dini condition, Hwang and S. Kim (2020) proved that there is a non-negative Green function $G_D^{\mathcal{L}}(x,y)$ on $D \times D \setminus$ diag of $\mathcal{L}$ on any bounded $C^{1,1}$ domain $D \subset \mathbb{R}^d$ in the sense that for any $f \in L^p(D)$ with $p > d/2$,

$$u(x) := G_D^{\mathcal{L}} f(x) = \int_D G_D^{\mathcal{L}}(x,y) f(y) dy$$

is in $W^{2,p}(D) \cap W_0^{1,p}(D)$ and is a strong solution for

$$\begin{cases} \mathcal{L}u = -f & \text{in} \quad D \\ u = 0 & \text{on} \quad \partial D. \end{cases}$$

- Furthermore, Hwang and Kim(2020) proved that

$$G_D^{\mathcal{L}}(x,y) \le C|x-y|^{2-d}$$

$$|\nabla_x G_D^{\mathcal{L}}(x,y)| \le C|x-y|^{1-d}$$

Moreover, if D is a bounded $C^{2,\text{Dini}}$ domain,

$$|\nabla_x^2 G_D^{\mathcal{L}}(x,y)| \le C|x-y|^{-d},$$

where $C = C(d, \lambda_0, R_0, \Lambda_0, \text{diam}(D))$.

Question:

What are the sharp estimates of $G_D^{\mathcal{L}}(x,y)$ on a bounded $C^{1,1}$ domain? What about the first and second derivative estimates?

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Our setting

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- Suppose $(a_{ij}(x))_{1\leq i,j \leq d}$ is a symmetric $d \times d$ matrix-valued function on $\mathbb{R}^d$ and there exists a constant $\lambda_0 \geq 1$ such that for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$,

$$\lambda_0^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \lambda_0 |\xi|^2.$$

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Our setting

- Since $A(x)$ is uniformly continuous, bounded and uniformly elliptic, the martingale problem for $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$ is well posed and its solution forms a conservative strong Markov process X that has continuous sample paths and strong Feller property.
- We call $G_D(x,y)$ is a Green function of X in D, or equivalently, of $\mathcal{L}$ on D, if for any non-negative $f \in C_c(D)$,

$$\mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \int_D G_D(x,y)f(y)dy \quad \text{for every } x \in D.$$

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Theorem (Chen-W., 2021):

Suppose D is a bounded $C^{1,1}$ domain with characteristics (R_0, Λ_0). There is a unique jointly continuous non-negative function $G_D(x,y) = G_D^{\mathcal{L}}(x,y)$ on $D \times D \setminus \text{diag}$ of the process X. Moreover, there exist $K_i = K_i(d, \lambda_0, \ell, R_0, \Lambda_0, \text{diam}(D)) > 1, i = 1, 2, 3$ such that

(i)

$$K_1^{-1} G_D^{\Delta}(x,y) \le G_D(x,y) \le K_1 G_D^{\Delta}(x,y);$$

(ii)

$$|\nabla_x G_D(x,y)| \le \frac{K_2}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right);$$

(iii)

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} G_D(x,y) \right| \le \frac{K_3}{|x-y|^d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{-1}.$$

Step 1: Identification of $G_D(x,y) = G_D^{\mathcal{L}}(x,y)$

By Hwang-Kim(2020), for each $f \in L^q(D)$ with $q > d$,

$$u(x) := G_D^{\mathcal{L}} f(x) = \int_D G_D^{\mathcal{L}}(x,y) f(y)\, dy$$

is in $W^{2,q}(D) \cap C(\overline{D})$ and is a strong solution of

$$\begin{cases} \mathcal{L}u &= -f \quad \text{in} \quad D; \\ u &= 0 \quad \text{on} \quad \partial D. \end{cases}$$

Let $f \in C_c(D)$ and $u(x) := G_D^{\mathcal{L}} f(x)$ Let $u_\varepsilon = u * \psi_\varepsilon$ be the mollifier of u. By Ito's formula,

$$\mathbb{E}_x \left[u_\varepsilon(X_{t \wedge \tau_{D_{\varepsilon_0}}}) \right] - u_\varepsilon(x) = \mathbb{E}_x \left[\int_0^{t \wedge \tau_{D_{\varepsilon_0}}} \mathcal{L} u_\varepsilon(X_s) ds \right] = \mathbb{E}_x \left[\int_0^{t \wedge \tau_{D_{\varepsilon_0}}} -f * \psi_\varepsilon(X_s) ds \right], x \in D_{\varepsilon_0}.$$

By taking $t \to \infty, \varepsilon \to 0$ and letting $\varepsilon_0 \to 0$,

$$G_D^{\mathcal{L}} f(x) = u(x) = \mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right], \quad x \in D.$$

Then

$$G_D(x, y) = G_D^{\mathcal{L}}(x, y).$$

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$$G_D(x,y) = G_D^{\mathcal{L}}(x,y).$$

Step 2: Levi's freezing coefficient method

Define for each fixed $z \in \mathbb{R}^d$,

$$\mathcal{L}^{(z)} := \sum_{i,j=1}^{d} a_{ij}(z) \frac{\partial^2}{\partial x_i \partial x_j}.$$

For each fixed $z \in \mathbb{R}^d$, denote by $G_D^{(z)}$ the Green function of $\mathcal{L}^{(z)}$ in D. We search for Green function $G_D(x,y)$ of $\mathcal{L}$ in D of the following form:

$$G_D(x,y) = G_D^{(y)}(x,y) + \int_D G_D^{(z)}(x,z) g_D(z,y) dz$$

for some function $g_D(x,y)$.

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Formally applying $\mathcal{L}$ on both sides in x, we have

$$\begin{aligned}
-\delta_{\{y\}}(x) &= \mathcal{L}^{(x)} G_D^{(y)}(\cdot,y)(x) + \mathcal{L}^{(x)} \int_D G_D^{(z)}(\cdot,z)(x) g_D(z,y) dy \\
&= \mathcal{L}^{(y)} G_D^{(y)}(\cdot,y)(x) + (\mathcal{L}^{(x)} - \mathcal{L}^{(y)}) G_D^{(y)}(\cdot,y)(x) \\
&\quad + \int_D (\mathcal{L}^{(z)} + (\mathcal{L}^{(x)} - \mathcal{L}^{(z)})) G_D^{(z)}(\cdot,z)(x) g_D(z,y) dy \\
&= -\delta_{\{y\}}(x) + (\mathcal{L}^{(x)} - \mathcal{L}^{(y)}) G_D^{(y)}(\cdot,y)(x) - g_D(x,y) \\
&\quad + \int_D (\mathcal{L}^{(x)} - \mathcal{L}^{(z)}) G_D^{(z)}(\cdot,z)(x) g_D(z,y) dy.
\end{aligned}$$

Here $\delta_{\{y\}}$ stands for the Dirac measure concentrated at y.

$$G_D(x,y) = G_D^{(y)}(x,y) + \int_D G_D^{(z)}(x,z) g_D(z,y) dz$$

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We obtain

$$\begin{aligned} g_D(x,y) =& (\mathcal{L}^{(x)} - \mathcal{L}^{(y)})G_D^{(y)}(\cdot,y)(x) \\ &+ \int_D (\mathcal{L}^{(x)} - \mathcal{L}^{(z)})G_D^{(z)}(\cdot,z)(x) g_D(z,y)dy. \end{aligned}$$

Setting

$$g_D^{(0)}(x,y) := (\mathcal{L}^{(x)} - \mathcal{L}^{(y)})G_D^{(y)}(\cdot,y)(x),$$

we see that $g_D(x,y)$ should satisfy the following integral equation

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Applying the above equation recursively, it motives us to define

$$g_D(x,y) := \sum_{k=0}^{\infty} g_D^{(k)}(x,y)$$

whenever it converges, where

$$g_D^{(k+1)}(x,y) := \int_D g_D^{(0)}(x,z) g_D^{(k)}(z,y) dz \quad \text{for } k \geq 0.$$

Lemma 1:

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exist constants $C_k = C_k(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D)) \geq 1$, $k = 1, 2, 3$, such that for every $\lambda > 0$, $z \in \mathbb{R}^d$ and $x \neq y$ in λD,

$$G^{(z)}_{\lambda D}(x, y) \asymp \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right);$$

$$|\nabla_x G^{(z)}_{\lambda D}(x, y)| \leq \frac{C_2}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right);$$

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G^{(z)}_{\lambda D}(x, y)\right| \leq \frac{C_3}{|x-y|^{d}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right)^{-1}.$$

Lemma 2:

There exist positive constants $\theta = \theta(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) \in (0,1)$ and $C = C(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $\lambda \in (0, \theta]$,

$$g_{\lambda D}(x,y) := \sum_{k=0}^{\infty} g_{\lambda D}^{(k)}(x,y)$$

converges absolutely and locally uniformly on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$ with

$$|g_{\lambda D}(x,y)| \le \sum_{k=0}^{\infty} |g_{\lambda D}^{(k)}(x,y)| \le C \frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \quad \text{for } x \neq y \in \lambda D.$$

Identification of the Levi's freezing formula

$$G_{\lambda D}(x,y) = G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)dz, \quad x \neq y \in \lambda D.$$

- Define
$$\widetilde{G}_{\lambda D}(x,y) := G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)dz, \quad x \neq y \in \lambda D.$$
- For each $C^\infty_c(\lambda D)$ function ψ, $\widetilde{G}_{\lambda D}\psi(x) := \int_{\lambda D} \widetilde{G}_{\lambda D}(x,y)\psi(y)dy$ is in $W^{2,p}_{loc}(\lambda D)$ for any $p > 1$ and
$$\begin{cases} \mathcal{L}\widetilde{G}_{\lambda D}\psi & = -\psi \quad \text{in} \quad \lambda D \\ \widetilde{G}_{\lambda D}\psi & = 0 \quad \text{on} \quad \partial(\lambda D). \end{cases}$$
- Once this is established, we can use mollifier and Ito's formula to show that $\widetilde{G}_{\lambda D}(x,y)$ is the occupation density $G_{\lambda D}(x,y)$ of the diffusion process X associated with $\mathcal{L}$ in λD.

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By Levi's freezing formula, we obtain

Lemma:

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exist positive constants $C = C(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ and $\theta = \theta(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that for any $\lambda \in (0, \theta]$,

$$G_{\lambda D}(x,y) \le \frac{C}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \quad \text{on } (\lambda D) \times (\lambda D) \setminus \mathrm{diag}$$

and for any $\gamma \ge 1$,

$$G_{\lambda D}(x,y) \ge \frac{C}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \quad \text{on } |x-y| \le \gamma \delta_{\lambda D}(x).$$

Green function estimates on small bounded $C^{1,1}$ domain

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Boundary decay rate of $G_D(x,y)$ in x

- For each subdomain V with $D \setminus \overline{V} \neq \emptyset$,

$$G_D(x,y) = \mathbb{E}_x\left[G_D(X_{\tau_V}, y); \tau_V < \tau_D\right] \quad \text{for every } x \in V \quad \text{and} \quad y \in D\setminus\overline{V}.$$

- By the test function argument in Chen-Kim-Song-Vondracek(2012) for $\Delta + \Delta^{\alpha/2}$, we obtain

Lemma:

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let $r_0 := \frac{R_0}{4\sqrt{1+\Lambda_0^2}}$. There are constants $\delta_0 = \delta_0(d, \lambda_0, R_0, \Lambda_0) \in (0, r_0)$ and $C_k = C_k(d, \lambda_0, R_0, \Lambda_0)$, $k = 1, 2$, such that for any $\lambda_1 > 1, Q \in \partial D$, and $x \in D_Q(\delta_0/\lambda_1, r_0/\lambda_1)$ with $\tilde{x} = 0$,

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\delta_0/\lambda_1, r_0/\lambda_1)}} \in D\right) \le C_1 \lambda_1 \delta_D(x),$$

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\delta_0/\lambda_1, r_0/\lambda_1)}} \in U_Q(\delta_0/\lambda_1, r_0/\lambda_1)\right) \ge C_2 \lambda_1 \delta_D(x).$$

Boundary decay rate of $G_D(x,y)$ in x

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Two-sided estimates of $G_{\lambda D}(x,y)$ on small domains

Theorem:

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exists a positive constant $C = C(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) \geq 1$ and $\theta = \theta(d, \lambda_0, R_0, \Lambda_0, \mathrm{diam}(D))$ such that for any $r \in (0, \theta]$,

$$C^{-1} G^{\Delta}_{\lambda D}(x,y) \leq G_{\lambda D}(x,y) \leq C G^{\Delta}_{\lambda D}(x,y) \quad \text{for } x \neq y \in \lambda D.$$

Two sided estimates of $G_D(x,y)$ in bounded $C^{1,1}$ domain D

Theorem:

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exists a positive constant $C = C(d, \lambda_0, \ell, \Lambda_0, R_0, \text{diam}(D)) \geq 1$

$$C^{-1} G_D^{\Delta}(x,y) \leq G_D(x,y) \leq C G_D^{\Delta}(x,y) \quad \text{for } x \neq y \in D.$$

Step 3: First and second derivatives of $G_D(x,y)$

- By taking derivative on both sides of Levi's freezing formula, we can obtain the upper bounded estimate of the first derivative $|\nabla_x G_D(x,y)|$.
- However, this method does not work well for the second order derivative estimates on $G_D(x,y)$. We use the second derivative estimate

$$|\nabla_x^2 G_B(x,y)| \le c|x-y|^{-d}$$

on balls from Hwang-Kim(2020) and an integral representation of the Green function of $\mathcal{L}$ on small balls to obtain the upper bound estimate of the second order derivative $|D_x^2 G_D(x,y)|$.

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Thank you for your attention!